CERN-TH/96-232
HUB-EP-96/49
hep-th/9609111

ON THE DUALITY BETWEEN THE HETEROTIC STRING AND F-THEORY IN 8 DIMENSIONS

Gabriel Lopes Cardoso[a], **Gottfried Curio**[b],
Dieter Lüst[b] and **Thomas Mohaupt**[b1]

[a] *Theory Division, CERN, CH-1211 Geneva 23, Switzerland*
[b] *Humboldt-Universität zu Berlin, Institut für Physik, D-10115 Berlin, Germany*

ABSTRACT

In this note we compare the moduli spaces of the heterotic string compactified on a two–torus and F–Theory compactified on an elliptic K3 surface for the case of an unbroken $E_8 \times E_8$ gauge group. The explicit map relating the deformation parameters α and β of the F–Theory K3 surface to the moduli T and U of the heterotic torus is found using the close relationship between the K3 discriminant and the discriminant of the Calabi–Yau–threefold $X_{1,1,2,8,12}(24)$ in the limit of a large base $\mathbf{P}^1$.

September 1996
CERN-TH/96-232

[1] Email: cardoso@surya11.cern.ch, curio@qft2.physik.hu-berlin.de, luest@qft1.physik.hu-berlin.de, mohaupt@qft2.physik.hu-berlin.de

During the last year it has become likely that a non–perturbative formulation of string theory (or of the theory behind it) requires the introduction of additional dimensions beyond the critical dimension 10 of perturbative supersymmetric string theories. From the space–time point of view all such constructions have in common that they enable one to regard the dilaton as a geometric modulus arizing from compactifying these extra dimensions.

There are at time two constructions of this type called M–theory [1] and F–theory [2] - [5]. More specifically, in [2] Vafa conjectured that II B superstring theory should be regarded as the toroidal compactification of twelve–dimensional F–theory. Adapting the stringy cosmic string construction [6] new compactifications of the II B strings on D–manifolds were constructed in which the complexified coupling varies over the internal space. These compactifications then have a beautiful geometric interpretation as compactifications of F–theory on elliptically fibred manifolds, where the fibre encodes the behaviour of the coupling, the base is the D–manifold, and the points where the fibre degenerates specifies the positions of the D–branes in it. Moreover compactifications of F–theory on elliptic Calabi–Yau twofolds (the K3), threefolds and fourfolds can be argued to be dual to certain heterotic string theories in 8,6 and 4 dimensions and have provided new insights into the relation between geometric singularities and perturbative as well as non–perturbative gauge symmetry enhancement and into the structure of moduli spaces.

The simplest but already instructive example of a F–theory compactification is given by the compactification to 8 dimensions on a K3 which is believed to be dual to the heterotic string on a two–torus. It is the purpose of this note to make explicit the precise relation between a specific part of the moduli spaces of both theories.

In the next section we will first consider the K3 discriminant with special emphasis on its singular points which are precisely related to the points of enhanced gauge symetries for particular values of the heterotic T_2 moduli. In the subsequent section we will work out the precise relation between the heterotic T_2 moduli T, U and the corresponding F-theory $K3$ moduli α, β. For this purpose we will use the four-dimensional duality [7] - [15] between the heterotic string on $T_2^{T,U} \times K3^H$ with special $SU(2)$ instanton embedding and the type IIA string on a Calabi–Yau-threefold CY. The relevant CY is a $K3$ fibration, i.e. locally we can write $CY = \mathbf{P}_1 \times K3^{CY}$. The four-dimensional duality between the heterotic string and F–theory on $CY \times T_2^F$ is then implied by following the arguments given in [2, 3], where the considered CY is at the same time an elliptic fibration. For concreteness we choose CY to be given by $X_{1,1,2,8,12}(24)$, with Hodge numbers $h_{1,1} = 3$ and $h_{2,1} = 243$. This leads to three vector moduli S, T and U, where all other gauge

symmetries are broken by generic values of the hyper multiplets.

Now we consider the decompactification limit to six dimensions by making the base $\mathbf{P}_1$ large, which is common to all three models (regarding $K3^H$ locally as $\mathbf{P}_1 \times T_2^{K3}$). In this way, using the (reverse) adiabatic argument of [16], we are dealing with a heterotic string on $T_2^{T,U} \times T_2^{K3}$ where the non-Abelian gauge symmetries $E_8 \times E_8$ are now broken by the Wilson line vector multiplets. The six-dimensional heterotic string is in turn dual to the type IIA string on $K3^{CY}$ or respectively dual to F–theory on $T_2^F \times K3^{CY}$. Now it is important to remember that, in the limit of large $\mathbf{P}_1$, the heterotic moduli T and U can by explicitly related to the $K3^{CY}$ moduli via the mirror map from IIA to IIB. Finally, we can trade T_2^F for T_2^{K3}, and we can directly compare the 8-dimensional heterotic string on $T_2^{T,U}$ (with non-vanishing Wilson lines and completely broken $E_8 \times E_8$ gauge symmetries) to F–theory on $K3^{CY}$. In this way we will obtain the exact relation between the heterotic moduli T, U and the $K3^{CY}$ moduli. Note that so far F–theory on $K3^{CY}$ has broken $E_8 \times E_8$ gauge symmetries. We will explain how we relate $K3^{CY}$ to a different $K3$ with $E_8 \times E_8$ singularities and hence unbroken $E_8 \times E_8$ gauge symmetries.

The K3 discriminant and its relation to heterotic moduli

Vafa has argued [2] that the heterotic string compactified on a two–torus in the presence of Wilson lines is dual to F–theory compactified on the family

$$y^2 = x^3 + f^{(8)}(z)x + f^{(12)}(z) \tag{1.1}$$

of elliptic K3 surfaces, where $f^{(k)}(z)$ is a polynomial of order $k = 8, 12$ respectively. In particular F–theory on the two parameter subfamily

$$y^2 = x^3 + \alpha z^4 x + (z^5 + \beta z^6 + z^7) \tag{1.2}$$

of K3's with E_8 singularities at $z = 0, \infty$ is dual to the heterotic theory with Wilson lines switched off [4]. Therefore there must exist a map which relates the complex structure and Kähler moduli U and T of the torus on which the heterotic theory is compactified to the two complex structure moduli α and β in (1.2). We will work out this map explicitly.

Let us first recall that it was claimed in [4] that in a certain limit the K3 fibre becomes constant over the base and moreover is then identical to the heterotic torus. Here "being identical" means having the same complex structure, since the Kähler modulus of the elliptic fibre of the K3 is frozen.

On the heterotic side the limit to be considered is the decompactification limit in which the Kähler modulus T is sent to infinity. On the F–theory side one sends both α and

β to infinity, keeping the ratio $\frac{\alpha^3}{\beta^2}$ fixed. This has the effect that the complex structure of the fibre becomes constant away from $z = 0, \infty$. To make this explicit note that the complex structure modulus τ_z of the fibre can be read off from the cubic equation (1.1) to be

$$j(\tau_z) = \frac{1728}{1 + \frac{27}{4}\frac{(f^{(12)}(z))^2}{(f^{(8)}(z))^3}}. \tag{1.3}$$

Using the special form (1.2) of the coefficents $f^{(k)}(z)$ and identifying the limit $\alpha, \beta \to \infty$, $\frac{\alpha^3}{\beta^2}$ finite with $T \to \infty$, U arbitrary, the prediction is that

$$j(iU) = \lim_{\alpha,\beta\to\infty} j(\tau_z) = \frac{1728}{1 + \frac{27}{4}\frac{\beta^2}{\alpha^3}}, \quad \text{if} \quad j(iT) = \infty. \tag{1.4}$$

We will show later on that this is indeed the case.

In order to get more information about the relation of (T, U) to (α, β) we now make use of the fact that for the heterotic string on a two–torus the generic gauge group $U(1)^2$ is enhanced to $SU(2)\times U(1)$, $SU(2)^2$ and $SU(3)$ for $T = U$, $T = U = 1$ and $T = U = e^{2\pi i/12}$ (neglecting the E_8^2 which is already present in ten dimensions and unbroken due to the absense of Wilson lines). In F–theory these gauge symmetry enhancements must arise from singularities of the K3 surface at special values of the parameters α, β. In our example these singularities must be of type A_1, A_1^2 and A_2 (neglecting the two E_8 singularities which are present for all values of α and β at $z = 0, \infty$).

Our next step is to compute at which values of α and β these singularities occur on the K3. To do so we start from the defining equation of the surface

$$F(x, y, z) = x^3 - y^2 + \alpha\, z^4\, x + z^5 + \beta z^6 + z^7 = 0 \tag{1.5}$$

and look for singularities by solving $F_x = F_y = F_z = 0$. Substituting these potential singular points back into the defining equation (1.5) gives a relation between the parameters which takes the form $\Delta(\alpha, \beta) = 0$, where $\Delta^{(K3)} = \Delta(\alpha, \beta)$ is the discriminant of the surface. In our case one finds

$$\Delta^{(K3)} = \left(\alpha^3 + \frac{27}{4}\beta^2 + 27\right)^2 - 27^2\beta^2, \tag{1.6}$$

which can be factorized as $\Delta^{(K3)} = \prod_i(\beta - \beta_i)$, where $\beta_{\pm1,\pm1} = \pm2 \pm \frac{2\sqrt{3}}{9}(-\alpha)^{3/2}$. As long as both α and β are not zero the four zeros of the discriminant are distinct. Along each of the four branches $\beta = \beta_{\pm1,\pm1}(\alpha)$ there is precisely one singular point (x_0, y_0, z_0) (outside $z = 0, \infty$) which is located at $\left(\epsilon_2\sqrt{-\frac{\alpha}{3}},\ 0,\ -\epsilon_1\right)$ for $\beta = \beta_{\epsilon_1,\epsilon_2}$ (,where $\epsilon_{1,2} = \pm1$).

Since the matrix of second derivatives is non–degenerate for $\alpha \neq 0$ ($\det(F_{ij}) \sim \sqrt{-\alpha}$) the singularity is of type A_1, because one can redefine coordinates such that the surface is

locally given by $F(x,y,z) = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = 0$ near the critical point, which is the standard form of an A_1 singularity.

Let us then discuss what singularities appear if two zeros of the discriminant coincide, starting with the case $\beta = 0$. This implies $\alpha^3 = -27$. Taking for example the solution $\alpha = -3$ we find that it corresponds to an intersection of the branches $\beta_{+1,-1}$ and $\beta_{-1,+1}$, which shows that there are simultanously two critical points located at $(-1,\ 0,\ -1)$ and $(1,\ 0,\ 1)$. Both are of type A_1 because $\det(F_{ij}) \sim \sqrt{-\alpha} \neq 0$. The same happens for the other two solutions $\alpha = 3(\frac{1}{2} \pm \frac{i}{2}\sqrt{3})$ of $\alpha^3 = -27$ with different localizations of the critical points.

The other case is $\alpha = 0$ which implies $\beta^2 = 4$. The two subcases $\beta = \pm 2$ correspond to intersections of branches $\beta_{+1,+1} = \beta_{+1,-1}$ and $\beta_{-1,+1} = \beta_{-1,-1}$. This time there is only one singular point located at $(0,\ 0,\ -1)$ for $\beta = 2$ and $(0,\ 0,\ 1)$ for $\beta = -2$. The singularity is not of type A_1, because $\alpha = 0 \Rightarrow \det(F_{ij}) = 0$. At the critical point (x_0, y_0, z_0) one computes $F_{xx} = 0$, $F_{xxx} \neq 0$, $F_{yy} \neq 0$, $F_{zz} \neq 0$, and $F_{ij} = 0$, if $i \neq j$. Therefore the surface can be locally brought to the form $F(x,y,z) = (x-x_0)^3 + (y-y_0)^2 + (z-z_0)^2 = 0$ making explicit that the singularity is of type A_2.

The appearence of surface singularities is related to the location of singular fibres as is well known from the stringy cosmic string construction [6]. In general the locations of degenerate fibres are obtained by solving

$$j(\tau_z) = 1728 \cdot \frac{4 \cdot (f^{(8)}(z))^3}{4 \cdot (f^{(8)}(z))^3 + 27 \cdot (f^{(12)}(z))^2} = \infty \tag{1.7}$$

for z. For generic moduli this is equivalent to solving

$$\Delta^{(T)} = 4 \cdot (f^{(8)}(z))^3 + 27 \cdot (f^{(12)}(z))^2 = 0, \tag{1.8}$$

where $\Delta^{(T)}$ is the discriminant of the elliptic fibre. In general this equation has 24 distinct solutions corresponding to 24 non–coinciding singular fibres. Restricting to the two–parameter family (1.2) and ignoring singular fibres over $z = 0, \infty$ this becomes

$$4\alpha^3 z^2 + 27(z^2 + \beta z + 1)^2 = 0. \tag{1.9}$$

For generic α, β there are four distinct roots corresponding to four singular fibres over four different points on the base. On the discriminant locus $\Delta(\alpha, \beta) = 0$ two of the four roots coincide , whereas at the A_1^2 point the four roots combine into two pairs. Finally at the A_2 point all four roots coincide. This gives a nice explicit example of how singularities come about in the stringy cosmic string construction of K3: whereas isolated singular fibres give regular points, all kinds of ADE singularities can be obtained by letting singular fibres coincide in a particular way [6].

The discriminant $\Delta^{(K3)}$ (1.6) of the K3 surface and the discriminant $\Delta^{(T)}$ of its elliptic fibre are closely related. One can check that $\Delta^{(K3)}$ is the discriminant in the usual (algebraic) sense of the discriminant $\Delta^{(T)}$ of the elliptic fibre: $\Delta^{(K3)}$ vanishes if two or more zeros of $\Delta^{(T)}$ coincide, reflecting the fact that K3 singularities do not come from singular fibres but from coinciding singular fibres. This is analogous to the relation between K3 fibred Calabi–Yau threefolds and their K3 fibres which were discussed in [9], [17]. Moreover there is a simple asymptotic relationship between the two discriminants in the limit $\alpha, \beta \to \infty$, $\frac{\alpha^3}{\beta^2}$ finite:

$$\frac{\Delta^{(K3)}}{\alpha^6} = \left(\frac{\Delta^{(T)}}{\alpha^3}\right)^2 + O\left(\frac{\beta^2}{\alpha^6}, \frac{1}{\alpha^3}\right). \tag{1.10}$$

Again a similar relation was observed in the Calabi–Yau context in [9], [17].

Summarizing we have found the critical values for α, β where the K3 surfaces (1.2) devellop A_1, A_1^2 or A_2 singularities. These values must be mapped to the corresponding critical values of the heterotic moduli T, U:

$$\begin{array}{lclcl} T = U & \leftrightarrow & j(iT) = j(iU) & \leftrightarrow & \Delta^{(K3)} = \Delta(\alpha,\beta) = 0 \\ T = U = 1 & \leftrightarrow & j(iT) = j(iU) = 1728 & \leftrightarrow & \alpha^3 = -27,\ \beta = 0 \\ T = U = e^{2\pi i/12} & \leftrightarrow & j(iT) = j(iU) = 0 & \leftrightarrow & \alpha = 0,\ \beta^2 = 4. \end{array} \tag{1.11}$$

We also expect to find the asymptotic relation (1.4) in the limit $j(iT) \to \infty$, $j(iU)$ finite corresponding to $\alpha, \beta \to \infty$, $\frac{\alpha^3}{\beta^2}$ finite. Finally note that on the heterotic side mirror symmetry exchanges T and U. Therefore one might expect that α and β are given by symmetric combinations $j(iT) \cdot j(iU)$ and $j(iT) + j(iU)$ as the case of the map between the (T, U) moduli space and the weak coupling limit of the conifold locus of the Calabi–Yau threefold $X_{1,1,2,8,12}(24)$ [7], [9]. In fact we can use that map in order to relate (T, U) to (α, β), as we will explain in the next section.

Relation to the S-T-U Calabi Yau

As argued in the introduction, one should be able to relate α and β to T and U, by using some of the results of [9] for the S-T-U Calabi–Yau $X_{1,1,2,8,12}(24)$. A priori, one might have thought that this isn't possible, because F–theory on K3 exhibits E_8–type singularities, whereas the S-T-U Calabi–Yau doesn't. The point, however, is that in F–theory there is a clear distinction between moduli of the $K3$, and moduli of the $E_8 \times E_8$.

The perturbative gauge symmetry enhancement on the heterotic side along $T = U$ is controlled by the "middle polynomials" xz^4 and z^6 on the F–theory side, whereas the "lower and higher polynomials" are related to enhancement to $E_8 \times E_8$ or a subgroup thereof. Thus, the idea is to first obtain the "middle polynomials" for the S-T-U Calabi–Yau $X_{1,1,2,8,12}(24)$, and then to identify them with the "middle polynomials" of F–theory compactified on $K3$. By doing so, one obtains an explicit map between the F–theory moduli α and β and the heterotic moduli T and U. Some of the discussion given on page 5 of [18] seems to be pointing into the same direction. The defining polynomial for the Calabi–Yau $X_{1,1,2,8,12}(24)$ is, according to eq. (7) of [9], given by

$$p = x_1^{24} + x_2^{24} + x_3^{12} + x_4^3 + x_5^2 - 12\psi_0 x_1 x_2 x_3 x_4 x_5 - 2\psi_1 (x_1 x_2 x_3)^6 - \psi_2 (x_1 x_2)^{12}. \quad (1.12)$$

In order to show that this Calabi–Yau is a $K3$–fibration, we set $x_2 = \lambda x_1$, $\tilde{x}_1 = x_1^2$ and obtain

$$p_{K3} = x_3^{12} + x_4^3 + x_5^2 + (1 + \lambda^{24} - \psi_2 \lambda^{12}) \tilde{x}_1{}^{12} - 12\psi_0 \lambda \tilde{x}_1 x_3 x_4 x_5 - 2\psi_1 \lambda^6 (\tilde{x}_1 x_3)^6. \quad (1.13)$$

This describes the $K3$ fiber of the Calabi–Yau $X_{1,1,2,8,12}(24)$. We would like to write it in the form of a Weierstrass equation, that is in the form $y^2 = x^3 + x f^{(8)}(z) + f^{(12)}(z)$. With $x_5 = y, x_4 = x, x_3 = z, \tilde{x}_1 = w$ and renaming $x \to -x$ the equation $p_{K3} = 0$ turns into

$$y^2 + (12\psi_0 \lambda z w) y x = x^3 - z^{12} - (1 + \lambda^{24} - \psi_2 \lambda^{12}) w^{12} + 2\psi_1 \lambda_6 (zw)^6. \quad (1.14)$$

This is precisely of the form of eq. (3.2) in [18], with the following identifications

$$\begin{aligned} a_1 &= 12\psi_0 \lambda z w \ , \quad a_2 = a_3 = a_4 = 0 \\ a_6 &= -z^{12} - (1 + \lambda^{24} - \psi_2 \lambda^{12}) w^{12} + 2\psi_1 \lambda^6 (zw)^6. \end{aligned} \quad (1.15)$$

The coefficients b_j of eq. (3.3) of [18] are then given by $b_2 = a_1^2, b_4 = 0, b_6 = 4a_6, b_8 = b_2 a_6$. The Weierstrass form can now be obtained by completing the square in y and then completing the cube in x. The resulting functions $f^{(8)}$ and $f^{(12)}$ are then given as follows (eq. (3.4) of [18])

$$\begin{aligned} f^{(8)} &= -\frac{1}{48} b_2^2 = -\frac{1}{48} a_1^4 = \alpha_{CY} z^4 w^4 \\ f^{(12)} &= \frac{1}{864}(b_2^3 + \frac{1}{216} b_6) = \frac{1}{864}(a_1^6 + 864 a_6) \\ &= -z^{12} - \gamma w^{12} + \beta_{CY} z^6 w^6. \end{aligned} \quad (1.16)$$

with $\alpha_{CY} = -432(\psi_0 \lambda)^4, \beta_{CY} = 3456(\psi_0 \lambda)^6 + 2\psi_1 \lambda^6$ and $\gamma = 1 + \lambda^{24} - \psi_2 \lambda^{12}$. Note that $f^{(8)}$ and $f^{(12)}$ have degrees 8 and 12, respectively, in z and w (in the following we will

work in the chart $w = 1$). Note that $f^{(8)}(z)$ contains precisely (and only) a z^4-term, whereas $f^{(12)}(z)$ contains a z^6-term, but no z^5 or z^7 term and, hence, no singularities of the E_8 type, as should be the case for the Calabi–Yau $X_{1,1,2,8,12}(24)$. Instead, $f^{(12)}(z)$ contains a constant z^{12}-term as well as a z^0-term. Next, consider rewriting $f^{(8)}(z)$ and $f^{(12)}(z)$ in terms of the complex structure moduli $\bar{x}$ and $\bar{z}$ (not to be confused with the earlier coordinates x, z), given in [9]: $\bar{x} = -\frac{\psi_1}{3456\psi_0^6}$, $\bar{z} = -\frac{\psi_2}{\psi_1^2}$.

Thus

$$\alpha_{CY}^3 = \frac{27}{4}\psi_2\lambda^{12}\frac{1}{\bar{x}^2\bar{z}}, \qquad \beta_{CY}^2 = -\psi_2\lambda^{12}\frac{(1-2\bar{x})^2}{\bar{x}^2\bar{z}}. \tag{1.17}$$

Now we consider the weak coupling limit $\psi_2 \to \infty$, i.e. the limit of large $\mathbf{P}_1$. With

$$\begin{aligned} \bar{x} &= 864\frac{j(iT)+j(iU)-1728}{j(iT)j(iU)+\sqrt{j(iT)(j(iT)-1728)}\sqrt{j(iU)(j(iU)-1728)}}, \\ \bar{z} &= 864^2\frac{1}{j(iT)j(iU)\bar{x}^2}, \end{aligned} \tag{1.18}$$

one finds, using the relation $4\bar{x}(1-\bar{x}) = 1728\frac{j(iT)+j(iU)-1728}{j(iT)j(iU)}$, for the combination

$$\frac{\beta_{CY}^2}{\alpha_{CY}^3} = -\frac{4}{27}(1-2\bar{x})^2 = -\frac{4}{27}(1-1728\frac{j(iT)+j(iU)-1728}{j(iT)j(iU)}). \tag{1.19}$$

So finally one has

$$\begin{aligned} j(iT)j(iU) &= \frac{1728^2}{\psi_2\lambda^{12}}\frac{\alpha_{CY}^3}{27}, \\ (j(iT)-1728)(j(iU)-1728) &= -\frac{1728^2}{\psi_2\lambda^{12}}\frac{\beta_{CY}^2}{4}. \end{aligned} \tag{1.20}$$

Let us now interpret the above result in the light of our original question. For this we have to go from the K3 in eq.(1.2) (in homogenized form),

$$y^2 = x^3 + \alpha z^4 w^4 x + z^5 w^7 + \beta z^6 w^6 + z^7 w^5, \tag{1.21}$$

to the K3 of the S-T-U Calabi–Yau, defined in eq.(1.16),

$$y^2 = x^3 + \alpha_{CY} z^4 w^4 x - \gamma w^{12} + \beta_{CY} z^6 w^6 - z^{12}, \tag{1.22}$$

by the redefinition $z \to \rho(z,w)z$, $w \to \sigma\frac{1}{\rho(z,w)}w$. This leads to the condition $\sigma^{12} = \gamma$ and

$$\alpha_{CY}^3 = \gamma\alpha^3, \quad \beta_{CY}^2 = \gamma\beta^2. \tag{1.23}$$

Replacing α_{CY}, β_{CY} by α, β corresponds to the transition from non-vanishing heterotic Wilson lines with broken $E_8 \times E_8$ to the case of vanishing Wilson lines with unbroken $E_8 \times E_8$, which was the starting point in eq.(1.2). So the discriminant for the locus $T = U$ we got from the $T, U \leftrightarrow \alpha_{CY}, \beta_{CY}$ matching above is given by

$$\begin{aligned}
(j(iT) - j(iU))^2 &\sim ((-\frac{1}{\psi_2\lambda^{12}})\frac{\alpha_{CY}^3}{27} + (-\frac{1}{\psi_2\lambda^{12}})\frac{\beta_{CY}^2}{4} - 1)^2 + 4(-\frac{1}{\psi_2\lambda^{12}})\frac{\alpha_{CY}^3}{27} \\
&= ((-\frac{1}{\psi_2\lambda^{12}})\frac{\alpha_{CY}^3}{27} + (-\frac{1}{\psi_2\lambda^{12}})\frac{\beta_{CY}^2}{4} + 1)^2 - 4(-\frac{1}{\psi_2\lambda^{12}})\frac{\beta_{CY}^2}{4} \\
&= ((-\frac{\gamma}{\psi_2\lambda^{12}})\frac{\alpha^3}{27} + (-\frac{\gamma}{\psi_2\lambda^{12}})\frac{\beta^2}{4} + 1)^2 - 4(-\frac{\gamma}{\psi_2\lambda^{12}})\frac{\beta^2}{4}.
\end{aligned} \tag{1.24}$$

This is precisely proportional to the discriminant eq.(1.6) found in the previous section as in the weak coupling limit $\psi_2 \to \infty$ one has $\gamma \to -\psi_2\lambda^{12}$.

In summary, let us display our main result which explicitly relates the moduli T and U of the heterotic string compactified on T_2 to the moduli α and β of F-theory on $K3$:

$$\begin{aligned}
j(iT)j(iU) &= -1728^2\frac{\alpha^3}{27}, \\
(j(iT) - 1728)(j(iU) - 1728) &= 1728^2\frac{\beta^2}{4}.
\end{aligned} \tag{1.25}$$

As as useful check of this results consider the ratio $\frac{\beta^2}{\alpha^3}$ in the limit $T \to \infty$:

$$\frac{\beta^2}{\alpha^3} \to -\frac{4}{27}(1 - \frac{1728}{j(iU)}). \tag{1.26}$$

This is in precise agreement with eq.(1.4), and we therefore find that in the limit $\alpha, \beta \to \infty$ and $\frac{\beta^2}{\alpha^3}$ fixed, the fibre modulus equals the heterotic modulus iU in accordance with [4].

Let us finally remark that our result can be regarded as the two-parameter generalization of the one-parameter torus $y^2 = x^3 + \alpha x + \beta$. Comparing this with the well-known Weierstrass form ($g_2 = \frac{4}{3}\pi^4 E_4$, $g_3 = \frac{8}{27}\pi^6 E_6$)

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau) \tag{1.27}$$

yields $\alpha = -\frac{1}{3}\pi^4 E_4$, $\beta = -\frac{2}{27}\pi^6 E_6$; so in this case one has with $\Delta = \frac{(2\pi)^{12}}{1728}(E_4^3 - E_6^2)$ that

$$\begin{aligned}
j(\tau)\Delta(\tau) &= -1728^2\frac{\alpha^3}{27}, \\
(j(\tau) - 1728)\Delta(\tau) &= 1728^2\frac{\beta^2}{4}.
\end{aligned} \tag{1.28}$$

References

[1] J. Schwarz, hep-th/9607201.

[2] C. Vafa, hep-th/9602022.

[3] D. R. Morrison and C. Vafa, hep-th/9602114.

[4] D. R. Morrison and C. Vafa, hep-th/9603161.

[5] A. Sen, hep-th/9605150.

[6] R. R. Greene, A. Shapere, C. Vafa and S. T. Yau, Nucl. Phys. **337** (1990) 1.

[7] S. Kachru and C. Vafa, Nucl. Phys. **B 450** (1995) 69, hep-th/9505105.

[8] V. Kaplunovsky, J. Louis and S. Theisen, Phys. Lett. **B 357** (1995) 71, hep-th/9506110.

[9] A. Klemm, W. Lerche and P. Mayr, Phys. Lett. **B 357** (1995) 313, hep-th/9506112.

[10] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, Nucl. Phys. **B 455** (1995) 109, hep-th/9507115.

[11] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, Nucl. Phys. **B 459** (1996) 537, hep-th/9508155.

[12] G. Curio, Phys. Lett. **B 366** (1996) 131, hep-th/9509042; Phys. Lett. **B 368** (1996) 78, hep-th/9509146.

[13] P.S. Aspinwall and J. Louis, Phys. Lett. **B 369** (1996) 233, hep-th/9510234.

[14] G. L. Cardoso, G. Curio, D. Lüst, T. Mohaupt and S.–J. Rey, Nucl. Phys. **B 464** (1996) 18, hep-th/9512129.

[15] G. L. Cardoso, G. Curio, D. Lüst and T. Mohaupt, hep-th/9603108, to appear in Phys. Lett **B**.

[16] C. Vafa and E. Witten, hep-th/9507050

[17] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, hep-th/9604034.

[18] M. Bershadsky, K. Intrilligator, S. Kachru, D. R. Morrison, V. Sadov and C. Vafa, hep-th/9605200.